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HIDDEN SPURIOUS SOURCES IN AXIAL GAUGE PROPAGATORS

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ABSTRACT

We show that charge carrying propagators in axial gauges involve spurious sources that move along the rays of gauge-fixing. Although these spurious sources are hidden in the axial gauge in question, they are manifest when the same propagator is viewed in other gauges. Therefore, they influence the propagation of the dynamical fields. Thus the naive electron propagator in axial gauge quantum electrodynamics does not have the spectrum of a free, mass-renormalized electron. We confine our remarks to quantum electrodynamics here. In the sequel the implications for axial gauge quark and gluon propagators in quantum chromodynamics is discussed. Gauge-independent propagators do not suffer this affliction.

I. Introduction

Since axial gauges were introduced by Schwinger [1] and Arnowitt and Fickler [2] much has been said about the interpretation of the "kinematical" singularities of the momentum space gauge-vector propagator [3]. Although these singularities do not occur in Green's functions of gauge-independent operators, they do affect gauge-dependent propagators. In this report we discuss a related axial gauge phenomenon that seems not to have been widely appreciated, namely, that in axial gauge the propagation of fields induced by charged operators occurs in the presence of spurious sources that, although hidden, nevertheless have a dynamical effect upon the propagation.

Our observation proceeds from a rather elementary remark that is easily illustrated in temporal axial gauge quantum electrodynamics. In this gauge physical states satisfy Gauss' law in the form

$$[\nabla\cdot\vec{E}(\vec{y}) - \rho(\vec{y})]|\mathrm{phys}\rangle = 0 \tag{1.1}$$

where $\rho(\vec{y}) = - e{:}\psi^{\dagger}(\vec{y})\,\psi(\vec{y}){:}$. Therefore, the state

$$\psi^{\dagger}(\vec{x})|\mathrm{phys}\rangle \tag{1.2}$$

satisfies a modified version of Gauss' law

$$[\nabla\cdot\vec{E}(\vec{y}) - \rho(\vec{y}) - e\delta^{3}(\vec{x}-\vec{y})]\ \psi^{\dagger}(\vec{x})|\mathrm{phys}\rangle = 0\ , \tag{1.3}$$

with an additional fixed "spurious" source at the position $\vec{x}$ with charge opposite the dynamical fermion. Apparently, all axial gauges have the feature that local gauge-dependent operators are associated with spurious point-like source currents. These currents affect the propagation of the dynamical field in peculiar ways. The appearance, interpretation, and dynamical effect of these sources is discussed at length in the following sections.

We shall confine our remarks to a class of "superaxial" gauges obtained by completely fixing the gauge up to a global gauge transformation. These are gauges in which any pair of space-time points x and y is connected by a unique path R defined by $r(\tau)$, where $r(0) = y$ and $r(1) = x$, such that

$$A^{\mu}(r) \frac{dr_{\mu}}{d\tau} = 0 \qquad (1.4)$$

for all r on the path. Thus, for example, we may define a super-temporal gauge by the conditions

$$\begin{aligned} &A^0(x) = 0 && \text{for all } x^{\mu} \\ &A^3(t^*, \vec{x}) = 0 && \text{for all } \vec{x} \\ &A^2(t^*,x^1,x^2,z^*) = 0 && \text{for all } x^1,x^2 \\ &A^1(t^*,x^1,y^*,z^*) = 0 && \text{for all } x^1 \quad . \end{aligned} \qquad (1.5)$$

where y^*, z^*, and t^* are real constants. Then every pair of points x and y is connected by a path that moves from x parallel to the x^0 axis to t^*, then parallel to the x^3 axis to z^*, then parallel to the x^2 axis to y^*, then parallel to the x^1 axis until it connects with a corresponding path from y. Retraced segments are then dropped leaving the unique path. The gauge given by (1.5) is simply a temporal axial gauge with a specific choice fixing the subsidiary gauge freedom that would otherwise have permitted arbitrary time-independent gauge transformations in temporal axial gauge. Fixing the gauge in this way removes any ambiguity in the propagators. It is necessary to remove any residual gauge freedom for another reason: Failing to do so in temporal gauge prevents propagation of charged operators between different points in three space [4].

After a brief discussion of the origin of the spurious sources in the functional integral formulation of quantum electrodynamics in Sec. II, we give an explicit, elementary example of their appearance in the classical Maxwell theory and in the one-loop electron self-energy in perturbative quantum electrodynamics in Sec. III. We mention some consequences in the concluding section.

II. Origin of the Spurious Sources

Green's functions for quantum electrodynamics are obtained from the usual generating functional

$$Z(J_\mu, \bar{\eta},\eta) = \int[dA_\mu][d\bar{\psi}][d\psi]$$

$$\exp i\,[W(A_\mu, \bar{\psi},\psi) - \int(J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi)d^4x] \;, \qquad (2.1)$$

where $W(A_\mu, \bar{\psi},\psi)$ is the QED action for the vector potential A_μ and Dirac fields ψ, $\bar{\psi}$. The super-axial gauge Green's functions are obtained by restricting the functional integration over A_μ to those configurations conforming to the gauge restrictions* and carrying out the usual functional differentiation of ℓnZ with respect to the various currents. The fermion correlation function is, as usual

$$\langle 0|\psi(x)\,\bar{\psi}(y)|0\rangle_{SA} = \int[dA_\mu]_{SA}[d\bar{\psi}][d\psi]$$

$$\exp i[W(A_\mu,\bar{\psi},\psi)]\;\psi(x)\,\bar{\psi}(y)/Z_{SA}(0,0,0) \qquad (2.2)$$

where SA refers to a restricted integration over fields satisfying the super axial gauge condition. Of course on the right side the product $\psi(x)\,\bar{\psi}(y)$ can be replaced by the gauge-invariant expression

$$\psi(x)\;C(x,y,R)\bar{\psi}(y) \qquad (2.3)$$

where

$$C(x,y,R) = \exp\,[ie \int_R A^\mu(r)dr_\mu] \qquad (2.4)$$

without altering the result, provided that the path R is the unique path in this

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*In temporal gauge it is essential to include what is called in hamiltonian language the projection onto states satisfying Gauss' law. In the action language (2.1) this restriction is accomplished by taking care that on at least one timelike surface A_0 is not set to zero, but is integrated functionally. In this sense there is a slight difference between the conditions (1.5) and the conditions defining the functional manifold.

gauge connecting x and y satisfying (1.4), i.e. along which $A_\mu(x)\, dr^\mu = 0$. With such a path, C = 1. In other words, the expression

$$<0|\ \psi(x)\ C(x,y,R)\ \overline{\psi}(y)|0>_{SA} \ = \qquad (2.5)$$

$$\int[dA_\mu]_{SA}[d\overline{\psi}][d\psi]\exp[iW(A_\mu,\ \overline{\psi},\ \psi)]\ \psi(x)C(x,y,R)\ \overline{\psi}(y)/Z_{SA}(0,0,0)$$

is the same as (2.2), but is gauge invariant. After carrying out a gauge transformation from the gauge SA to a different gauge G the expression is still unchanged if the string remains at R. In the new gauge the string operator is not trivial. In this way the string "remembers" the gauge in which the correlation function was originally defined. However, in the new gauge the string corresponds to an explicit spurious external source that moves along the path R connecting x to y, i.e. the string operator has the form

$$C(x,y,R) = \exp\ (i\int J_s{}^\mu(x)\ A_\mu(x)\ d^4x] \qquad (2.6)$$

where

$$J^\mu{}_s(x) \ = e \int_0^1 d\tau\ \delta^4[x - r(\tau)]\ \frac{dr^\mu}{d\tau} \ , \qquad (2.7)$$

and where the path $r(\tau)$ is described in Sec. 1. The expression in (2.6), when substituted into (2.5), corresponds to an action in the presence of the external source $J_s{}^\mu(x)$. This external source certainly affects the propagation of the fermion even though it is hidden in the original gauge. Moreover, it contributes to the ultraviolet-divergent self energy of the propagator.

To be more concrete, consider the super-temporal gauge (1.5). Suppose that the point y does not coincide with the subsidiary gauge-fixing coordinates, i.e. $y^0 \neq t^*$, $y^3 \neq z^*$, $y^2 \neq y^*$, as is usually the case. Suppose, also, that $t^* > y^0$ and $x^0 > y^0$ so that a fermion is created at y and propagates to x. Then the path R emerges from y moving initially forwards and parallel to the

x^0 axis--i.e. it is initially a static source with charge opposite that of the fermion. Thus in operator language, the fermion creation operator, acting upon the physical vacuum, creates not only the dynamical fermion, but also generates a fixed opposite charge at the point of creation as noted in Sec. 1. As the dynamical fermion moves away from the point of creation it generates an electric field with flux lines that end on the fixed charge. Thus it propagates as an electron in the hydrogen atom in the approximation of an infinite proton mass and zero proton spin! The spectrum of the propagator must reflect the presence of the spurious source.

In $A^3 = 0$ axial gauge the current would run initially along the x^3 axis to a point z^*. In operator language such a current is associated with the creation of an infinitesimal tube of electric flux along the same path. The creation of such a structure also has a dramatic effect upon the fermion propagation.

The spurious sources associated with the vector potential are more subtle. Let us consider temporal axial gauge. It is convenient to consider the string-bit operator

$$B(\vec{x},d\vec{x}) = \exp\,[iq \int_{\vec{x}}^{\vec{x}+d\vec{x}} A_\mu(y)\, dy^\mu] \ , \tag{2.8}$$

for which

$$[\nabla\cdot\vec{E}(\vec{y}) - \rho(\vec{y}) + q\delta^3(\vec{x}+d\vec{x}-\vec{y}) - q\delta^3(\vec{x}-\vec{y})]\, B(\vec{x},d\vec{x})|\mathrm{phys}\rangle = 0 \ . \tag{2.9}$$

Therefore the operator B generates a pair of spurious sources at $\vec{x}$ and $\vec{x}+d\vec{x}$ with fixed charge strength q. The choice of charge strength here is entirely arbitrary, of course.

Since the Hilbert space of states containing fixed sources is orthogonal to the physical Hilbert space, which contains no fixed sources, it follows that for $q \neq 0$

$$\langle \text{phys}|B(\vec{x},\vec{dx})|\text{phys}\rangle = 0 \ . \qquad (2.10)$$

Consequently A_μ is an infinite operator on the physical sector. To see this, let us suppose that A_μ has finite matrix elements. Then in the limit $\vec{dx} \to 0$ we can approximate the exponential

$$\lim_{\vec{dx} \to 0} \langle \text{phys}|(1 + ie\,\vec{A}(x) \cdot \vec{dx})|\text{phys}\rangle = 0 \ . \qquad (2.11)$$

Therefore, as long as the physical state has non-zero norm, $\vec{A}(\vec{x})$ must have an expectation value that is not bounded from below. Failing to recognize this fact leads to bizarre consequences [5]. Other axial gauges undoubtedly have similar problems. To define the temporal axial gauge correlation function of the vector potential, naively given by

$$D_{ij}(x,y) = \langle 0|A_i(x)A_j(y)|0\rangle \ , \qquad (2.12)$$

we propose instead the expression

$$\lim_{\vec{dx},\vec{dy} \to 0} \langle 0|B^*(\vec{x},\vec{dx})\ B(\vec{y},\vec{dy})|0\rangle_{SA} = 1 + q^2\ dx_i dy_j D_{ij}(x,y) \qquad (2.13)$$

Of course the expression (2.12) is singular in temporal axial gauge, but the expression (2.13) does not appear to suffer from this difficulty.

Because the spurious sources induced by the vector potential correspond to an electric dipole of vanishing strength, they are not expected to have an effect upon the propagation of the photon in the limit $\vec{dx}$, $\vec{dy} \to 0$.

III. Spurious Sources in Classical and Quantum Electrodynamics

In this section we will demonstrate our previous remarks concerning spurious sources by performing two model calculations in electrodynamics. We will begin in a classical context by defining the photon propagator as the Green's function for the four-dimensional vector wave equation. This Green's function will reflect the boundary conditions imposed on the vector potential both without and with the subsidiary gauge constraints allowed in temporal axial gauge. In the classical theory, one deals with conserved current sources. However, in the quantum theory, because one often deals with non-gauge invariant operators, the photons can couple to non-conserved currents. Hence, with an eye towards the quantum theory, we will discuss the action of the classical Green's function on both conserved and non-conserved currents. Then we will proceed to a quantum description and calculate the second-order contribution to the electron propagator in temporal gauge QED. It will be shown that the super-temporal gauge electron propagator is equivalent to the Coulomb gauge propagator <u>in the presence of a fixed source</u>, which exists upon the rays of gauge fixing. Then we will relate this result to the gauge invariant electron propagator $G(x,y)$, defined by

$$G(x,y) \equiv \langle T(\psi(x)\, e^{ie\int_R dz_\mu A^\mu(z)}\, \overline{\psi}(y))\rangle \ , \qquad (3.1)$$

where the path R connects the points x and y according to the gauge condition.

A. Classical Theory

Our model calculation is to show how spurious source currents appear when we attempt to find the vector potential $A^\mu(x)$ due to some known current configuration $J^\mu(x)$. The equation of motion for the potential is given by

$$L^\mu{}_\nu\, A^\nu(x) \equiv [\Box\,\delta^\mu{}_\nu - \partial^\mu\partial_\nu]A^\nu(x) = J^\mu(x) \ . \qquad (3.2)$$

Because this equation is invariant under the change of gauge $A^\mu(x) \to A^\mu(x) + \partial^\mu\Lambda(x)$, $\Lambda(x)$ an arbitrary function of x, in order to solve it for A^μ we must first set constraints or gauge conditions upon A^μ. Having done this, we may solve (3.2) by finding the Green's function $D^\mu{}_\nu(x)$, such that

$$A^\mu(x) = \int d^4y \; D^\mu{}_\nu(x,y) \; J^\nu(x) \tag{3.3}$$

satisfies equation (3.2).

For example, in the Coulomb gauge we require that $\nabla\cdot\vec{A}_C = 0$ and that the potential $A^\mu{}_C(x)$ vanish at infinity. Then the Green's function is given by

$$\begin{aligned} D^\mu_{C\nu}(x,y) &= \delta^\mu{}_0\delta^0{}_\nu \quad \frac{\delta(t_x - t_y)}{4\pi|\vec{x} - \vec{y}|} \\ &- \delta^\mu{}_i \; \delta^j{}_\nu \; [\delta^i{}_j \; D_F(x-y) - \partial^i{}_x \; \partial_{yj} \int d^3x' \; \frac{D_F(t_x-t_y, \; \vec{x}'-\vec{y})}{4\pi|\vec{x} - \vec{x}'|} \;] \end{aligned} \tag{3.4}$$

[Greek indices run from 0 to 3, Latin indices from 1 to 3, and $\delta^\mu{}_i = 0$ if $\mu = 0$, $\delta^\mu{}_i = 1$ if $\mu = i = 1, 2,$ or 3.] Here $D_F(x-y)$ is the massless Feynman propagation function satisfying $\Box D_F(x-y) = \delta^4(x-y)$.

One may easily check, using $D^\mu{}_{C\nu}$ in (3.3), that

$$\begin{aligned} L^\mu{}_\nu \; A^\nu{}_C(x) &= J^\mu(x) + \delta^\mu{}_i \; \partial^i \int d^4y \; \frac{\delta(t_x-t_y)}{4\pi|\vec{x} - \vec{y}|} \; \partial_\nu J^\nu(y) \\ &= J^\mu(x) \; , \end{aligned} \tag{3.5}$$

if J^μ is a conserved current. This is consistent with (3.2), since $\partial_\mu \; L^\mu{}_\nu$ is a nilpotent operator. However, eq. (3.3) in itself makes no reference to conserved currents, and we may consider the action of $D_C{}^\mu{}_\nu$ upon non-conserved currents J^μ. In the case that $\partial_\mu J^\mu \neq 0$, we may define the right hand side of Eq. (3.5) to be a new current

$$J'^{\mu} = J^{\mu} + J^{\mu}_{s} \tag{3.6}$$

and find that $\partial_{\mu} J'^{\mu} = 0$. In any gauge this procedure defines a gauge dependent "return current" or spurious source that completes the non-conserved current. We are particularly interested here in the spurious sources associated with axial gauges.

Now let's turn to the temporal axial gauge $A^{o} = 0$. We may construct the temporal axial gauge potentials from the Coulomb gauge potentials by a gauge transformation:

$$A^{\mu}_{A}(x) = G^{\mu}_{AC\ \lambda}\ A^{\lambda}_{C}(x) \tag{3.7}$$

The subscripts A and C refer to axial and Coulomb, respectively, and $G^{\mu}{}_{\lambda}$ is a linear integral operator. For example, we may choose

$$G^{\mu}_{AC\lambda}(x,y) = \delta^{\mu}_{\lambda}\ \delta^{4}(x-y) - \partial^{\mu}_{x}\ \delta^{o}_{\lambda}\ \delta^{3}(\vec{x}-\vec{y})\ \theta(t_{x} - t_{y})\ \theta(t_{y}-T) \tag{3.8}$$

and then

$$A^{\mu}_{A}(x) = A^{\mu}_{C}(x) - \partial^{\mu} \int_{T}^{t_{x}} dt'\ A_{C}{}^{o}(t,x)\ . \tag{3.9}$$

Here T is an arbitrary end point. (The principal value prescription corresponds to a slightly different choice and is discussed after Eq. (3.19) below.)

Under the transformation (3.7) and (3.8), the Green's function changes according to

$$\begin{aligned} D^{\mu}_{A\nu}\ (x,y) &= \int d^{4}x'\ d^{4}y'\ G^{\mu}_{AC\lambda}(x,x')\ G^{\rho}_{AC\nu}(y,y')\ D^{\lambda}_{C\rho}(x',y') \\ &= \delta^{\mu}_{i}\ \delta^{j}_{\nu}\ [D^{i}_{Cj}(x,y) + \partial^{i}_{x}\ \partial_{yj} \int_{T}^{t_{x}} dt \int_{T}^{t_{y}} dt'\ D^{o}_{Co}(t,\vec{x};t',\vec{y})] \end{aligned} \tag{3.10}$$

Notice that in the new gauge the operator D_{A} is still the identity on the space of conserved currents. This can be seen by direct substitution

$$
\begin{aligned}
L_{x\ \nu}^{\ \mu} &\int d^4y\ D_A{}^\nu{}_\lambda(x,y)\ J^\lambda(y) \\
&= x^\mu{}_\nu \int d^4y\ d^4y'\ G_{AC}{}^\sigma{}_\lambda(y,y')\ D_C{}^\nu{}_\sigma(x,y')\ J^\lambda(y) \\
&= x^\mu{}_\nu \int d^4y\ D^\nu_{C\lambda}(x,y)\ J^\lambda(y) \\
&= J^\lambda(y) \qquad (3.11)
\end{aligned}
$$

where in obtaining the second line we use the fact that $L\partial$ is nilpotent, in obtaining the third line we integrated by parts and used $\partial\cdot J = 0$, and in the last line we used (3.5). The result is obviously true of any gauge.

As we have stated before, the $A^o = 0$ gauge constraint still allows for time independent gauge transformations. To completely specify the gauge in the manner of eqs. (1.5) we carry out further gauge transformations. Thus to fix $A^3(t^*, \vec{x}) = 0$, with t^* being an arbitrary fixed point in time, we define

$$
\begin{aligned}
A^\mu_{SA}(x) &= G^\mu_{SA\lambda}\ A^\lambda_A(x) \\
&= A^\mu_A(x) + \partial^\mu \int_{z^*}^{x^3} dx^{3'}\ A^3_A\ (t^*,\vec{x}')\ . \qquad (3.11)
\end{aligned}
$$

$[\vec{x}' \equiv (x^1,\ x^2,\ x^{3'}) \equiv (\vec{x}_\perp,\ x^{3'})]$. Here z^* is an arbitrary fixed point on the x^3-axis.

This transformation yields $D^\mu_{SA\nu}$ in terms of $D^\mu_{A\nu}$, and we get

$$
D^\mu_{SA\nu}(x,y) = \delta^\mu_i\ \delta^j_\nu\ [D^i_{Aj}(x,y) + \partial^i_x \int_{z^*}^{x^3} dx^{3'}\ D^3_{Aj}(t^*,\ \vec{x}';\ y)
$$

$$
- \partial_{yj} \int_{z^*}^{y^3} dy^{3'}\ D^i_{A3}(x;\ t^*,\ \vec{y}') - \partial^i_x\ \partial_{yj} \int_{z^*}^{x^3} dx^{3'} \int_{z^*}^{y^3} dy^{3'}\ D^3_{A3}(t^*,\ \vec{x}';\ t^*,\ \vec{y}')]. \qquad (3.12)
$$

Of course, at this point the gauge is still not completely specified for we are allowed to make transformations which are t and x^3 independent, as in eqs. (1.5). However, doing so would increase the number of terms in (3.12) to sixty-four and certainly not add constructively to our arguments. In the following we will avoid these complications by working only in the t - x^3 plane, keeping $\vec{x}_\perp$ fixed in all quantities. Afterwards we will comment briefly on the general case.

We shall now find the spurious current J_s that completes a non-conserved current in this super-temporal gauge. In analogy with the Coulomb gauge result (3.5) we consider the action of the operator D_{SA} upon various currents. As before this operator is the identity when acting upon conserved currents. Thus

$$L_x{}^\mu{}_\nu \int d^4y\ D_{SA}{}^\nu{}_\lambda(x,y)\ J^\lambda(y) = J^\mu(x) \qquad (3.12)$$

for $J^\mu(x)$ conserved. To find its action upon a non-conserved current we may proceed by a tedious direct evaluation in analogy with (3.5) or we may simply observe that if it is possible to guess a J_s such that $J + J_s$ is a conserved current, and such that $D_{SA}\ J_s$ vanishes, then we have the answer immediately, for then

$$\begin{aligned} &L_x{}^\mu{}_\nu \int d^4y\ D_{SA}{}^\nu{}_\lambda(x,y)\ J^\lambda(y) \\ &= \ {}_x{}^\mu{}_\nu \int d^4y\ D_{SA}{}^\nu{}_\lambda(x,y)\ [J^\lambda(y) + J_s{}^\lambda(y)] \\ &= J^\mu(y) + J_s{}^\mu(y)\ . \end{aligned} \qquad (3.13)$$

With our choice of subsidiary gauge fixing it is easy to find the desired J_s. For example let J^μ be a non-conserved line current

$$J^\mu(x) = \int_0^1 d\tau\ \delta^4[x - x_s(\tau)]\ \frac{dx_s{}^\mu}{dt} \qquad (3.14)$$

with $x_s(\tau)$ a trajectory in the x^0 - x^3 plane such that

$$x_s(0) = (\tau_1,0,0,z_1)$$

$$x_s(1) = (\tau_2,0,0,z_2) \ , \qquad (3.15)$$

as shown in Figure 1. Clearly

$$\partial_\mu \, J^\mu(x) = - \, \delta(t_x - \tau_2) \, \delta(x^3 - z_2) \, \delta^2(\vec{x}_\perp)$$

$$+ \, \delta(t_x - \tau_1) \, \delta(x^3 - z_1) \, \delta^2(\vec{x}_\perp) \quad . \qquad (3.16)$$

The return current J_s shown in Fig. 1 allows for current conservaton: i.e. with the definitions

$$J_s{}^\mu = \delta_3^\mu \, \theta(z_2 - x^3) \, \theta(x^3 - z_1) \, \delta(t_x - t^*) \, \delta^2(\vec{x}_\perp)$$

$$- \, \delta_0{}^\mu \, \theta(\tau_2 - t_x) \, \theta(t_x - t^*) \, \delta(x^3 - z_2) \, \delta^2(\vec{x}_\perp)$$

$$+ \, \delta_0^\mu \, \theta(\tau_1 - t_x) \, \theta(t_x - t^*) \, \delta(x^3 - z_1) \, \delta^2(\vec{x}_\perp) \ , \qquad (3.17)$$

the total current is conserved

$$\partial_\mu \, [J^\mu + J^\mu{}_s] = 0 \ . \qquad (3.18)$$

Furthermore, because J_s lies precisely along the rays of gauge fixing we have

$$\int d^4y \; D_{SA}{}^\nu{}_\lambda(x,y) \; J_s^\lambda(y) = 0 \ . \qquad (3.19)$$

Notice, of course, that we have chosen an example for which the return current occurs entirely in the $x^0 - x^3$ plane. With complete subsidiary gauge fixing a more general case could be considered, and the return current would follow the path R described in Section 1. Notice also, that the arbitrary constant T in (3.9) does not appear in (3.17). In fact, because we have chosen to construct the axial gauge quantities by starting in Coulomb gauge, we have implicitly chosen at each stage, a particular restriction on the remaining residual gauge freedom. This restriction is expressed in an indirect way in terms of the Coulomb gauge quantities. By replacing this implicit restriction by an explicit and simple choice, as in (1.5) through a sequence of gauge transformations as in (3.7) we remove by stages the dependence upon arbitrary constants, such as T.

The principle value prescription [3] corresponds to making the replacement $\int_T^{t_x} dt' \rightarrow \frac{1}{2}\int_{-\infty}^{\infty} dt' \; \varepsilon(t_x - t)$ in (3.9). If we follow up with the gauge transformation G_{SA} that makes A^3 vanish at t*, then we arrive at the same vector potential and the same spurious current as before.

It is interesting to speculate on the consequences of instead letting the Coulomb gauge potential implicitly fix the subsidiary gauge freedom, as in (3.9). In that case the spurious current associated with J^μ in (3.14) runs parallel to the time axis from τ_1 to T, then emerges from z_1 along electric dipole field lines at fixed t = T, converging on z_2, and then returns along a line parallel to the time axis from T to τ_2. With the principal value prescription replacing (3.9) as described above, the spurious source carries half the current to $t = +\infty$ and half to $t = -\infty$ at fixed z_1, and returns each half at fixed z_2. In any case, the spurious current is required.

We are led to the physical picture that $D_{SA}^{\mu\nu}$ contains contributions that do not couple to conserved currents but that do couple to non-conserved currents in a way that generates additional, spurious sources $J_s{}^\mu$ to yield a net conserved current. That is,

$$A_{SA}^{\mu}(x) = \int d^4y \; D_{SA}^{\mu}{}_{\rho}(x,y) \; J^{\rho}(y)$$

$$= \int d^4y \; D_{A\rho}^{\mu}(x,y) \; [J^{\rho}(y) + J_s^{\rho}(y)] + \partial^{\mu}\Lambda(x) \; . \qquad (3.20)$$

Here $\Lambda(x)$ is the function that takes A^μ from the temporal axial gauge to the superaxial gauge, and

$$\text{(i)}\quad J_s = 0 \qquad \text{for } J^\rho \text{ conserved} \;,$$

$$\text{(ii)}\quad \partial_\rho\,(J^\rho + J_s^\rho) = 0 \quad \text{otherwise} \;. \tag{3.21}$$

Then note, since $J + J_s$ is in any case a conserved current, (3.20) has a simple transformation law under a change of gauge. Suppose we write (3.20) in the Coulomb gauge:

$$D_C^{\mu\nu}(x,y) = \int dx'dy'\; G^\mu_{CA\lambda}(x,x')\; G^\nu_{CA\rho}(y,y')\; D_A^{\lambda\rho}(x',y') \tag{3.20}$$

Then

$$A^\mu_{SA}(x) = \int d^4y\; D^\mu_{C\rho}(x,y)\; [J^\rho(y) + J^\rho_s(y)] + \partial^\mu\Lambda'(x) \;, \tag{3.23}$$

where $\Lambda'(x)$ takes A^μ from the Coulomb gauge to the super-axial gauge. The same current appears in the integrand, but now $D^\mu_{C\rho}(x,y)$ does not vanish along the trajectory of J_s. In an axial gauge we may calculate a gauge invariant function and "hide" the spurious sources by forcing them to run along the rays of gauge fixing. In another gauge the spurious currents are manifest.

B. Perturbation Theory

We now extend our analysis to a quantum mechanical model. Specifically, we will use the action functional in quantum electrodynamics (QED) to find a photon propagation function identical to the one appearing in (3.12), and use it to show how string functions naturally appear in super-temporal gauge QED.

Consider the action or generating functional for the free photon propagator:

$$Z_{SA}\,[J] = \int [dA_\mu]_{SA}\; e^{i\{W[A] + J\cdot A\}} \tag{3.24}$$

(The notation $J \cdot A$ implies integration over space and time.) The subscript SA refers to having restricted the integration domain to include only those field configurations that satisfy the super-temporal gauge constraints (1.5). In (3.7) above we constructed the axial gauge from the Coulomb gauge by a gauge transformation linear in the field variables:

$$A^{\mu}_{A}(x) = G^{\mu}_{AC\lambda}\, A^{\lambda}_{C}(x) \qquad (3.25)$$

Since the Jacobian of this transformation is independent of A^{μ}, it does not contribute to functional derivatives of ln Z[J], and hence we may write

$$Z_A[J] = \int [dA_{\mu}]_C\, e^{i\{W[A] + J \cdot GA\}}\ , \qquad (3.26)$$

where we have indicated our restriction to Coulomb gauge fields by the subscript C. Thus, if D_C is the Coulomb gauge propagator, using an obvious notation,

$$Z_A[J] = \exp iJ \cdot [\vec{G} D_C \overleftarrow{G}] \cdot J\ . \qquad (3.27)$$

We conclude that the bare quantum propagator for the photon is $D^{\mu}_{A\nu}(x,y) = \int dx'dy'\, G^{\mu}_{AC\lambda}(x,x')\, G^{\rho}_{AC\nu}(y,y')\, D^{\lambda}_{C\rho}(x',y')$ as in (3.10). Since we may compound gauge transformations, we may follow the transformation (3.25) with the transformation (3.11) leading to the same expression for the propagator as the classical expression (3.12), provided we restrict our attention to the $x^0 - x^3$ plane.

Now consider the electron propagation function in super-axial gauge. In perturbation theory, we define

$$iS_{SA}(x,y) = iS^{0}_{F}(x-y) + iS^{(2)}(x,y) + O(\alpha^2)\ , \qquad (3.28)$$

where $iS^{(2)}(x,y)$ is the lowest-order gauge variant contribution to the propagator and is given by

$$iS^{(2)}(x,y) = -e^2 \int d^4z d^4z' \; S_F^0 \,(x-z)\; \gamma_\mu S_F^0(z-z')\; \gamma_\nu S_F^0(z'-y)\; D_{SA}^{\mu\nu}\,(z,z'), \quad (3.29)$$

and where $D_{SA}^{\mu\nu}(z,z')$ is given by (3.12).

Instead of proceeding directly, let us decompose $D_{SA}^{\mu\nu}(x,y)$ in terms of the Coulomb gauge propagator as follows:

$$\begin{aligned} D_{SA}^{\mu\nu}(x,y) &= D_C^{\mu\nu}(x,y) + [D_A^{\mu\nu}(x.y) - D_C^{\mu\nu}(x,y)] + [D_{SA}^{\mu\nu}(x,y) - D_A^{\mu\nu}(x,y)] \\ &\equiv D_C^{\mu\nu}(x,y) + \Delta D^{\mu\nu}(x,y) + \Delta\tilde{D}^{\mu\nu}(x,y) \;. \end{aligned} \quad (3.30)$$

$D_A^{\mu\nu}$ is given by (3.10), and $D_C^{\mu\nu}$ is given by (3.4). Note that

$$\begin{aligned} \Delta D^{\mu\nu}(x,y) &= [\delta_0^\mu \,\delta_0^\nu \,\partial_x^0 \,\partial_y^0 - \delta_i^\mu \,\delta_j^\nu \,\partial_x^i \,\partial_y^i] \int_T^{t_x} dt \int_T^{t_y} dt' \; D_C^{00}(x,y) \\ &= [-\,\partial_x^\mu \,\partial_y^\nu + (\delta_0^\nu \,\partial_x^\mu \,\partial_y^0 + \delta_0^\mu \,\partial_x^0 \,\partial_y^\nu)] \int_T^{t_x} dt \int_T^{t_y} dt' \; D_C^{00}(x,y) \;. \end{aligned} \quad (3.31)$$

A quick glance back at (3.12) shows that $\Delta\tilde{D}^{\mu\nu}$ can also be written in terms of single and double gradients, and thus we have the generic form

$$\begin{aligned} \Delta^{\mu\nu}(x,y) \equiv \Delta D^{\mu\nu}(x,y) + \Delta\tilde{D}^{\mu 0}(x,y) &= \partial_x^{\mu} g_1^\nu(x,y) \\ &+ \partial_y^\nu \, g_2^\mu \,(x,y) + \partial_x^\mu \,\partial_y^\nu \, h(x,y) \;, \end{aligned} \quad (3.32)$$

which is as it should be, since $D_{SA}^{\mu\nu}$ is related to $D_C^{\mu\nu}$ by a gauge transformation.

Now consider the amplitude that D_{SA} couples to in (3.29):

$$J_{dyn}^{\mu\nu}(x,y;\ z,z') = \overset{0}{S_F}(x-z)\ \gamma^\mu\ \overset{0}{S_F}\ (z-z')\ \gamma^\nu\ \overset{0}{S_F}(z'-y)\ . \qquad (3.33)$$

This amplitude is not conserved in either z or z', for a simple application of the Ward-Takahashi identity,

$$\partial_z^\mu\ [\overset{0}{S_F}(x-z)\ \gamma_\mu\ \overset{0}{S_F}(z-y)] = i\delta^4(x-z)\ \overset{0}{S_F}(z-y) - i\delta^4(z-y)\ \overset{0}{S_F}(x-z)\ , \qquad (3.34)$$

readily shows that $\partial_z^\mu\ J_{\mu\nu}(x,y;\ z,z') \neq 0$ and $\partial_{z'}^\nu\ J_{\mu\nu}(x,y;\ z,z') \neq 0$.

However, we may add contributions to $J_{dyn}^{\mu\nu}$ due to fixed spurious currents that run along trajectories where we had required $D_{SA}^{\mu\nu}$ to vanish by our subsidiary gauge transformations and in so doing obtain a conserved amplitude. Without changing $S^{(2)}(x,y)$, we could then replace

$$J_{dyn}^{\mu\nu}(x,y,z,z') \to J_{tot}^{\mu\nu}(x,y,z,z')$$

in (3.29) where

$$J_{tot}^{\mu\nu}(x,y;z,z') = J_{dyn}^{\mu\nu}(x,y;z,z') + J_{sp}^{\mu\nu}\ (x,y;z,z')\ . \qquad (3.35)$$

Such a spurious amplitude is

$$\begin{aligned} J_{sp}^{\mu\nu}(x,y;z,z') = \frac{1}{2}\ [&\overset{0}{S_F}(x-z)\ \gamma^\mu\ \overset{0}{S_F}(z-y) i\ J_s{}^\nu(z') \\ &+ \overset{0}{S_F}(x-z')\ \gamma^\nu\ \overset{0}{S_F}(z'-y) i\ J_s{}^\mu(z) \\ &- \overset{0}{S_F}(x-y)\ J_s{}^\mu(z)\ J_s{}^\nu(z')] \end{aligned} \qquad (3.36)$$

where J_s is given by (3.17). One may readily verify that

$$\partial_{z\mu} J^{\mu\nu}_{tot}(x,y,z,z') = \partial_{z'\nu} J^{\mu\nu}_{tot}(x,y;z,z') = 0. \tag{3.37}$$

That is, J_{tot} is a conserved amplitude. Thus we rewrite (3.37) as

$$\begin{aligned} iS^{(2)}(x,y) &= - e^2 \int d^4z d^4z' \; J^{\mu\nu}_{tot}(x,y,z,z') \, D_{SA\mu\nu}(z,z') \\ &= - e^2 \int d^4z d^4z' \; J^{\mu\nu}_{tot}(x,y,z,z') \, [D_{C\mu\nu}(z,z') + \Delta_{\mu\nu}(z,z')] \end{aligned} \tag{3.38}$$

Now using (3.33), integrating by parts, and using (3.37) we see that the remainder involving $\Delta_{\mu\nu}$ gives no contribution so that

$$i\, S^{(2)}(x,y) = - e^2 \int d^4z d^4z' \; J^{\mu\nu}_{tot}\,(x,y,z,z') \, D_{C\mu\nu}(z,z') \tag{3.39}$$

An examination of the form of $J^{\mu\nu}_{tot}$ reveals that the first order super-axial gauge self energy (3.39), now expressed in the language of Coulomb gauge, has the following graphical interpretation. The contribution $J^{\mu\nu}_{dyn}$ comes from the first order Coulomb gauge self energy in Fig. 2(a). The first and second terms in $J^{\mu\nu}_{sp}$ in (3.36) come from the interaction between the dynamical and spurious sources, shown as in Fig. 2(b), and the third term in $J^{\mu\nu}_{sp}$ comes from the self energy of the spurious source as in Fig. 2(c).

The reader may verify that (3.38) is precisely the form one gets when

$$iS_{SA}(x,y) = \langle T(\psi(x)\, e^{i\int d^4z\, A_\mu(z)\, J^\mu_s(z)}\, \overline{\psi}(y))\rangle \tag{3.40}$$

is evaluated to order e^2 in perturbation theory; i.e. with $J^\mu_s(z)$ specified in eq. (3.17), then we have the gauge invariant expression

$$iS_{SA}(x,y) = \langle T(\psi(x) e^{ie\int_R dz_\mu\, A^\mu(z)}\, \overline{\psi}(y))\rangle\ . \tag{3.41}$$

The line integral is evaluated along the trajectory R running from y to x described in Sec. 1.

IV. Concluding Remarks

We have shown that in quantum electrodynamics propagators of charged operators in axial gauge contain hidden spurious sources that affect the propagation of the dynamical fields. If the operators are also local, then they contribute to the ultraviolet divergent self-energy of the fields. We have given an explicit demonstration of the appearance of spurious sources in classical electrodynamics and in the electron propagator in QED. Similar problems arise in non-Abelian gauge theories. There the spurious sources may combine with the dynamical fields to produce a confined gauge singlet state. The consequences for non-Abelian theories are discussed in the sequel in the language of the Polyakov-Wilson lattice gauge theory [6].

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References

[1] J. Schwinger, Phys. Rev. 130 (1963) 402.

[2] R.L. Arnowitt and S.I. Fickler, Phys. Rev. 127 (1962) 1821.

[3] W. Kummer, Acta Phys. Austriaca, Suppl. XV (1976) 423.
W. Konetschny and W. Kummer, Nucl. Phys. B 100 (1975) 106.
S. Caracciolo, G. Curci, P. Menotti, Phys. Lett. 113B (1982) 311.
G.B. West, Phys. Rev. D27 (1983) 1878.

[4] S. Mandelstam, Phys. Rev. D19 (1979) 2391.

[5] Yujiro Kakudo, Yukio Taguchi, Azuma Tanaka, and Kunio Yamamoto, Phys. Rev. D27 (1983) 1954.

[6] C.E. DeTar, J.E. King, S.P. Li, and L. McLerran, University of Utah report UU HEP 84/7.

Figure Captions

1. Spurious current $J_S(x)$ induced by a non-conserved current $J(x)$ in a super-temporal gauge. In this gauge A^0 vanishes everywhere and A^3 vanishes at $x^0 = t^*$.

2. Coulomb gauge Feynman graphs for the $O(e^2)$ electron self energy showing the interaction with the spurious source x.

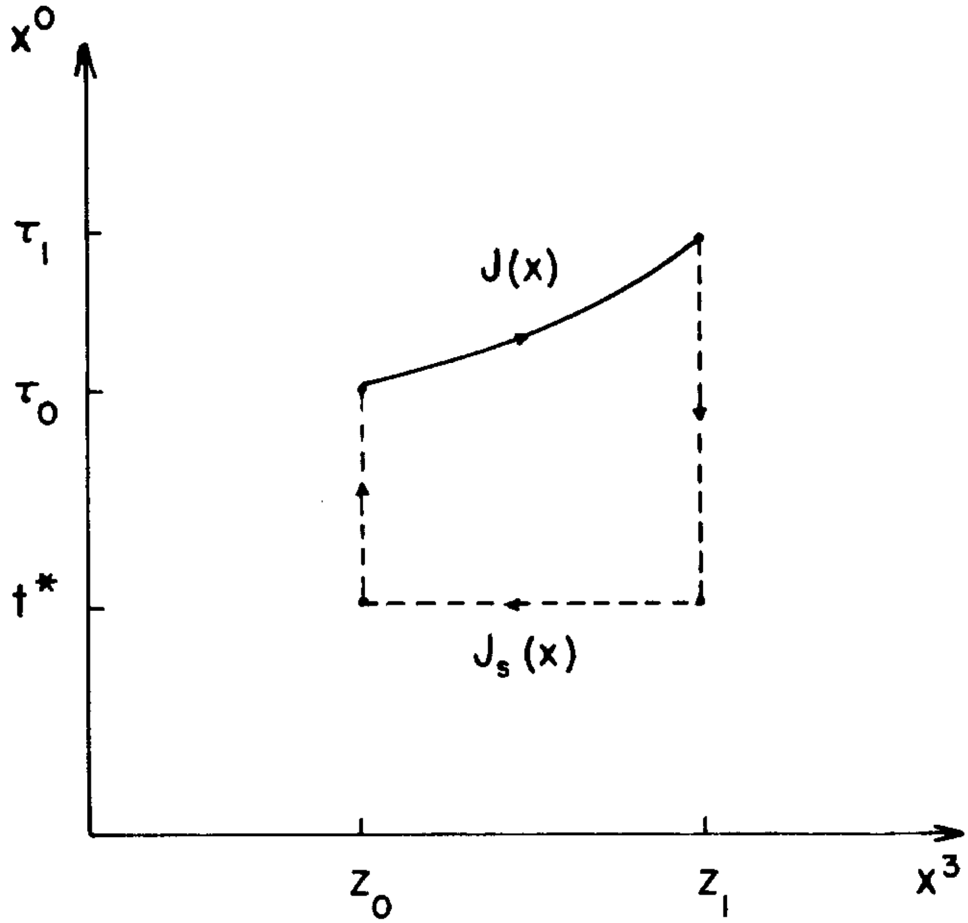

Figure 1

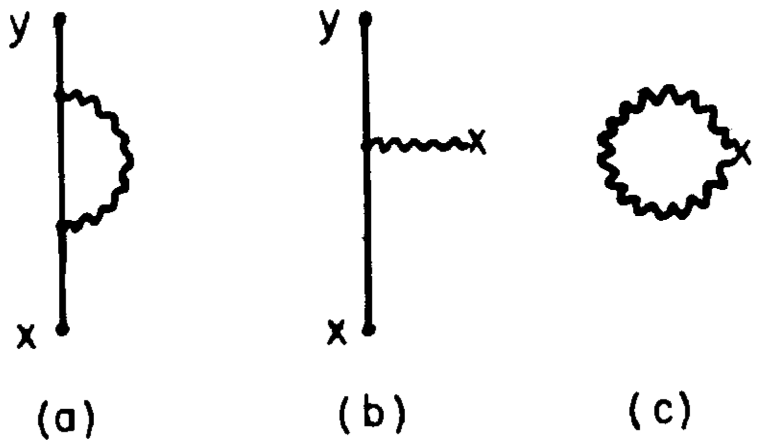

Figure 2